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On the Series of Sturm and Liouville, as Derived from a Pair of Fundamental Integral Equations Instead of a Differential Equation

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XI. *On the Series of* Sturm *and* Liouville, *as Derived from a Pair of Fundamental Integral Equations instead of a Differential Equation.*

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Introduction.

The series of Liouville and Sturm are generally treated by means of approximate solutions of the fundamental differential equation, these approximations being valid when certain functions involved in the differential equation have differential coefficients. The object of the present paper is to relax this restriction, and for this purpose integral equations are used in place of a differential equation, and an approximation is investigated (§§ 4–11) depending on a function which is constant throughout each of a system of sub-intervals.

In §§ 15–18 the results are applied, by help of Hobson's general convergence theorem, to that one of the Liouville series which is usually valid at the two ends of the fundamental interval, and in §§ 19–22 to the more general series discussed by me in 'Proc. L.M.S.,' ser. 2, vol. 3, pp. 83–103.

A theorem analogous to that of Vallée-Poussin on the series of squares of the Fourier constants is then proved (§§ 23–25) by a method which I believe to be new.

1. The differential equation of Liouville and Sturm is

$$\frac{d}{dx}\left(k\frac{dV}{dx}\right)+(gr-l)\,V = 0 \quad \ldots \ldots \ldots \quad (1)$$

and is equivalent to the pair of integral equations

$$U = \int V\,(l-gr)\,dx, \qquad V = \int \frac{1}{k}\,U\,dx. \quad \ldots \ldots \ldots \quad (2)$$

The values of U, V at the lower limit in these integrals are the two arbitrary constants of the complete primitive.

In the theory of the equation (1) the known functions k, g and the unknown function V are generally supposed to have differential coefficients; no such assumption will be made in the present treatment of the equations (2). All integrals will be taken according to Lebesgue. Also g, k are supposed positive and l real, and it is assumed that the integrals of g, k^{-1}, $|l|$ exist.

2. The equations (2) are somewhat simplified if we take $\int (g/k)^{1/2}\,dx$ as independent variable, and with a change of notation they become

$$\phi = \int \rho\Phi\,dx, \qquad \Phi = \int (\sigma - \lambda/\rho)\,\phi\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Here σ, ρ are known functions of x, ρ being positive, and it is assumed that the integrals of $|\sigma|$, ρ, $\frac{1}{\rho}$ exist; λ is a parameter independent of x, and our first object will be to find an approximate solution when $|\lambda|$ is great, but λ not necessarily real.

3. Values of ϕ, Φ satisfying (3) and such that, when $x = a$, $\phi = 0$, and $\Phi = 1$ will be denoted by $\phi(x, a)$, $\Phi(x, a)$; if, when $x = a$, $\phi = 1$, and $\Phi = 0$ they will be denoted by $\psi(x, a)$, $\Psi(x, a)$. If $\phi = A$, $\Phi = B$, when $x = a$, then the equations (3) become

$$\phi = A + \int_a^x \rho\Phi\,dx, \qquad \Phi = B + \int_a^x (\sigma - \lambda/\rho)\,\phi\,dx,$$

and have, according to the known theory of integral equations, a unique solution which must be

$$\phi = A\psi(x, a) + B\phi(x, a),$$

$$\Phi = A\Psi(x, a) + B\Phi(x, a).$$

Thus, it follows that

$$\phi(x, b) = \phi(a, b)\,\psi(x, a) + \Phi(a, b)\,\phi(x, a),$$

$$\psi(x, b) = \psi(a, b)\,\psi(x, a) + \Psi(a, b)\,\phi(x, a),$$

$$\Phi(x, b) = \phi(a, b)\,\Psi(x, a) + \Phi(a, b)\,\Phi(x, a),$$

$$\Psi(x, b) = \psi(a, b)\,\Psi(x, a) + \Psi(a, b)\,\Phi(x, a).$$

Other important relations are

$$\phi(x, b)\,\phi(a, c) - \phi(x, c)\,\phi(a, b) = \phi(x, a)\,\phi(b, c),$$

$$\psi(x, b)\,\psi(a, c) - \psi(x, c)\,\psi(a, b) = \phi(x, a)\,\Psi(c, b),$$

and so on.

4. If a second pair of equations of the type (3) is taken where ρ', σ', λ', ϕ', Φ' take the places of ρ, σ, λ, ϕ, Φ, we have

$$\phi\Phi' = \int \{\rho\Phi\Phi' + (\sigma' - \lambda'/\rho')\,\phi\phi'\}\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

by the formula for integration by parts, and similarly

$$\phi'\Phi = \int \{\rho'\Phi\Phi' + (\sigma - \lambda/\rho)\,\phi\phi'\}\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4')$$

so that

$$\phi\Phi'-\phi'\Phi = \int\{(\rho-\rho')\,\Phi\Phi'+(\sigma'-\sigma+\lambda/\rho-\lambda'/\rho')\,\phi\phi'\}\,dx. \quad . \quad . \quad . \quad . \qquad (5)$$

These integrals are indefinite, each carrying with it an additive constant to be determined by trial of some particular value of x. The formula (5) includes a great variety of particular results of which many are well known in the theory of the equation (1).

For instance, take ϕ, ϕ' to be $\phi(x, a)$, $\phi(x, b)$ so that $\rho = \rho'$, $\lambda = \lambda'$, $\sigma = \sigma'$; by putting $x = a, b$ in turn we have

$$\phi(b, a) = -\phi(a, b),$$

and similarly

$$\psi(b, a) = \Phi(a, b), \qquad \Psi(b, a) = -\Psi(a, b),$$

and

$$\psi(x, a)\,\Phi(x, a)-\phi(x, a)\,\Psi(x, a) = 1.$$

Again, taking ϕ, ϕ' to be $\phi(x, a)$, $\phi'(x, b)$, we have

$$\phi(b, a)+\phi'(a, b) = \int_a^b\{(\rho-\rho')\,\Phi(x, a)\,\Phi'(x, b)+(\sigma'-\sigma+\lambda/\rho-\lambda'/\rho')\,\phi(x, a)\,\phi'(x, b)\}\,dx \quad (6)$$

which is the formula to be used in the approximation. The right side of (6) is the error committed when $\phi'(b, a)$ is taken as equal to $\phi(b, a)$; in it the term depending on $\sigma'-\sigma$ turns out to be unimportant, while when $\lambda' = \lambda$ the rest is numerically less than the square root of

$$\int_a^b(\rho-\rho')^2\,dx\times\int_a^b\left|\Phi(x, a)\,\Phi'(x, b)-\frac{\lambda}{\rho\rho'}\,\phi(x,a)\,\phi'(x,b)\right|^2 dx.$$

It will be our object to choose ρ' so that $\int_a^b(\rho-\rho')^2\,dx$ is small, while at the same time ϕ' can be expressed in terms of known functions, and, in fact, the interval (a, b) will be divided into small sub-intervals in each of which ρ' will be constant. Thus we need the following lemma:—

If $f(x)$ is a function limited and summable in (a, b) this interval can be so divided into a finite number of sub-intervals, and a function $\phi(x)$ constant in each sub-interval can be so chosen that $\int_a^b(fx-\phi x)^2\,dx$ is arbitrarily small.

5. To prove the lemma, let U, L be the boundaries of $f(x)$; take $n-1$ arithmetic means $a_1, a_2, \ldots, a_{n-1}$ between them and let $a_0 = \mathrm{L}$, $a_n = \mathrm{U}$. Let the values in (a, b) for which $a_{r-1} < f(x) \leq a_r$ form the set l_r $(r = 0, 1, 2, \ldots, n)$. Enclose l_r in a set of intervals Δ_r, not overlapping, and $\mathrm{C}(l_r)$ in a set Γ_r, not overlapping, so that Δ_r, Γ_r have a common part $< \epsilon$. Let the intervals of Δ_r in descending order of length be $\delta_{r1}, \delta_{r2}, \ldots$, and take an integer p such that

$$\sum_{m=1}^{p}\delta_{r,m} > \Delta_r-\epsilon\,(r = 0, 1, 2, \ldots, n).$$

Let $\epsilon_{r,m}$ be the part of $\delta_{r,m}$ which is also in Γ_r.

Put

$$\phi(x) = a_0 \text{ in } \delta_{0,m} (m = 1, 2, \ldots, p),$$
$$= a_1 \text{ in } \delta_{1,m} (m = 1, 2, \ldots, p),$$

except where the value a_0 has been already assigned

$$= \ldots = a_r \text{ in } \delta_{r,m} (m = 1, 2, \ldots, p),$$

except at such points as belong to $\delta_{0,m}, \delta_{1,m}, \ldots, \delta_{r-1,m} (m = 1, \ldots, p)$ where the value has been already assigned, and, lastly,

$$= \text{L in the rest of the domain.}$$

Then $|f(x) - \phi(x)| < (\text{U}-\text{L})/n$ in all parts of $\delta_{r,m}$ that are not in

$$\Gamma_r (r = 0, 1, \ldots, n; \ m = 1, 2, \ldots, p),$$

that is in intervals E whose sum is

$$\geqq \sum_{r=0}^{n} \sum_{m=1}^{p} (\delta_{rm} - \epsilon_{rm}).$$

Now

$$\sum_{r=0}^{n} \sum_{m=1}^{p} \delta_{r,m} > \sum_{r=0}^{n} \Delta_r - (n+1)\epsilon > (b-a) - (n+1)\epsilon,$$

$$\sum_{r=0}^{n} \sum_{m=1}^{p} \epsilon_{r,m} < (n+1)\epsilon,$$

and, therefore,

$$\text{E} > (b-a) - 2(n+1)\epsilon.$$

Hence the points where $|f(x) - \phi(x)| > (\text{U}-\text{L})/n$ form a set whose measure $< 2(n+1)\epsilon$, and

$$\int_a^b \{f(x) - \phi(x)\}^2 dx < (b-a)(\text{U}-\text{L})^2/n^2 + (\text{U}-\text{L})^2\, 2(n+1)\epsilon.$$

In this expression we may suppose $\epsilon = n^{-2}$ and make n as great as we wish so that the whole is arbitrarily small.

The most advantageous value for $\phi(x)$ in each interval is the average of $f(x)$ over the interval, for when c is to be a constant in the interval (x_0, x_1) the least value of

$$\int_{x_0}^{x_1} (fx - c)^2 dx$$

is given by putting

$$c = \int_{x_0}^{x_1} f(x)\, dx \div (x_1 - x_0).$$

6. The proof that has been given indicates a particular method of subdivision, say the method A, but any other method, say B, may also be used. To prove this, let

$\alpha_1, \alpha_2, \ldots$ be the points of a subdivision A_1, according to A, for which the value of the integral is $< \epsilon$.

Take a subdivision B_1, according to B, in which the sum of the intervals containing $\alpha_1, \alpha_2, \ldots, \alpha_n$ is $< \epsilon$: then in the other intervals of B_1 the value of ϕx is constant whether in A_1 or B_1 and the integral can therefore be made $< \epsilon$, while for the intervals containing $\alpha_1, \ldots, \alpha_n$ the value of the integral is not more than $(U-L)^2 \epsilon$. Hence for the subdivision B_1 the integral can be made less than

$$\epsilon + (U-L)^2 \epsilon,$$

that is, arbitrarily small.

Hence the subdivisions of (a, b) may be taken all equal, or according to any other method, provided that the greatest of them tends to zero.

Moreover, the square of

$$\int_a^b | fx - \phi x | \, dx$$

is less than

$$(b-a) \int_a^b (fx - \phi x)^2 \, dx,$$

and, therefore,

$$\int_a^b | fx - \phi x | \, dx$$

is also made arbitrarily small.

If L is positive, then

$$\int_a^b \left| \frac{1}{fx} - \frac{1}{\phi x} \right| dx < \frac{1}{L^2} \int_a^b | fx - \phi x | \, dx,$$

and is also arbitrarily small. We may, in fact, say that

$$\int_a^b | fx - \phi x | \, dx + \int_a^b \left| \frac{1}{fx} - \frac{1}{\phi x} \right| dx$$

is arbitrarily small.

This is the property that will be immediately useful. It may be extended to an unlimited function fx when

$$\int_a^b | fx | \, dx \qquad \text{and} \qquad \int_a^b \left| \frac{1}{fx} \right| dx$$

exist. For take a limited function $f_1 x$ which is equal to fx when $N \geq (fx)^2 \geq 1/N$ N being a certain positive number, and is 1 for other values of x. N may be so chosen that

$$\int_a^b | fx - f_1 x | \, dx \qquad \text{and} \qquad \int_a^b \left| \frac{1}{fx} - \frac{1}{f_1 x} \right| dx \qquad \text{both} < \tfrac{1}{4}\epsilon.$$

Then the function ϕx, constant in each of a system of sub-intervals, can be so determined that

$$\int_a^b \left\{ | f_1 x - \phi x | + \left| \frac{1}{f_1 x} - \frac{1}{\phi x} \right| \right\} dx < \tfrac{1}{2}\epsilon.$$

Then, by help of the inequality,

$$|a-b|+|b-c| \geq |a-c|,$$

it follows that

$$\int_a^b \left\{ |fx-\phi x| + \left| \frac{1}{fx} - \frac{1}{\phi x} \right| \right\} dx < \epsilon.$$

In the same way, if

$$\int_a^b (fx)^2 \, dx$$

exists, ϕ may be so determined that

$$\int_a^b (fx-\phi x)^2 \, dx < \epsilon,$$

by help of the inequality

$$(a-b)^2+(b-c)^2 \geq \tfrac{1}{2}(a-c)^2.$$

7. Thus, for the purpose of approximation, we are to divide the domain of x into sub-intervals, all tending to zero, and in each sub-interval put for ρ a suitable mean among its values over the sub-interval, which may be conveniently called a local average of ρ and denoted by r. Suppose, then, that

$$u(x, a) = u = \int_a^x r\mathrm{U}\, dx, \qquad \mathrm{U}(x, a) = \mathrm{U} = 1 - \int_a^x \frac{\lambda u}{r}\, dx,$$

$$v(x, a) = v = 1 + \int_a^x r\mathrm{V}\, dx, \qquad \mathrm{V}(x, a) = \mathrm{V} = -\int_a^x \frac{\lambda v}{r}\, dx.$$

In each sub-interval u, v, U, V are solutions of the equation

$$\frac{d^2y}{dx^2} = -\lambda y,$$

and are, therefore, of the form

$$\mathrm{A} \exp x\sqrt{-\lambda} + \mathrm{B} \exp(-x\sqrt{-\lambda}).$$

A, B are constants through each sub-interval, but are changed at the passage from one to another. It is of great importance to ascertain whether they can increase or decrease indefinitely, and we shall now prove that they cannot if the total fluctuation of $\log r$ is uniformly limited,* that is, if the total fluctuation is always less than a fixed finite quantity at all stages of subdivision of the domain.

8. At every internal point of a sub-interval v, V have differential coefficients, and at the points of division their derivatives, upper and lower, on the same side are equal.

* This does not imply that the total fluctuation of $\log \rho$ must be limited. For instance, ρ may be 1 at all rational points and 2 at irrational points, then $r = 2$ everywhere.

Since

$$\frac{dv}{dx} = r\mathrm{V} = -r\int_a^x \frac{\lambda v}{r}\, dx,$$

and the integral is a continuous function of x, it follows that the discontinuity in $\log\dfrac{dv}{dx}$ is equal to that in $\log r$. Also v has no discontinuity, being equal to $1+\displaystyle\int_a^x r\mathrm{V}\, dx$.

Let the imaginary quantities conjugate to λ, v, ... be denoted by $\bar{\lambda}$, $\bar{v}$, ..., and let $w = v\bar{v}$, $\sqrt{-\lambda} = \alpha+\iota\beta$, α being positive. Also write D for d/dx. Then in each sub-interval

$$(\mathrm{D}^2-4\alpha^2)(\mathrm{D}^2+4\beta^2)\, w = 0.$$

Also,

$$\begin{aligned}(\mathrm{D}^2+4\beta^2)\, w &= v\mathrm{D}^2\bar{v}+\bar{v}\mathrm{D}^2 v+2\mathrm{D}v\mathrm{D}\bar{v}+4\beta^2 v\bar{v},\\ &= 2(\alpha^2+\beta^2)\, v\bar{v}+2\mathrm{D}v\mathrm{D}\bar{v},\\ \mathrm{D}(\mathrm{D}^2+4\beta^2)\, w &= 4\alpha\,\{(\alpha+\iota\beta)\, v\mathrm{D}\bar{v}+(\alpha-\iota\beta)\,\bar{v}\mathrm{D}v\}.\end{aligned}$$

Hence, when $x = a$, the value of $(\mathrm{D}^2+4\beta^2)\, w$ is $2(\alpha^2+\beta^2)$, or $2|\lambda|$, and that of its derivative is 0. At any discontinuity when r is changed to r' (both are real) the derivative is multiplied by r'/r and $(\mathrm{D}^2+4\beta^2)\, w$ itself consists of two positive terms of which the first is unchanged while the second is multiplied by $(r'/r)^2$. Thus the effect on $(\mathrm{D}^2+4\beta^2)\, w$ is to multiply it by a quantity between 1 and $(r'/r)^2$.

Now, if in any interval a function y satisfies the differential equation $\mathrm{D}^2 y = 4\alpha^2 y$, and if at the beginning of that interval y lies between $\mathrm{A}\cosh 2\alpha(x-a)$ and $\mathrm{B}\cosh 2\alpha(x-a)$, while $\mathrm{D}y$ lies between $2\mathrm{A}\alpha\sinh 2\alpha(x-a)$ and $2\mathrm{B}\alpha\sinh 2\alpha(x-a)$ where A, B, $x-a$ are real and positive, then these same statements must hold good throughout the interval. For suppose $\mathrm{B} > \mathrm{A}$, then $\mathrm{B}\cosh 2\alpha(x-a)-y$ and $y-\mathrm{A}\cosh 2\alpha(x-a)$ are both functions satisfying the equation

$$\mathrm{D}^2 y = 4\alpha^2 y,$$

and at the beginning of the interval their values and derivatives are all positive; it follows from the differential equation that the values and derivatives will increase, and therefore be positive throughout the interval.

In the first sub-interval $(\mathrm{D}^2+4\beta^2)\, w$ must be $2|\lambda|\cosh 2\alpha(x-a)$ and its derivative $4\alpha|\lambda|\sinh 2\alpha(x-a)$. At entrance to any later sub-interval of the domain (a, b) $(a < b)$ this function and its derivative are multiplied by quantities of which we know that each lies between 1 and τ^2 where τ is the ratio of increase in r. Thus throughout all the sub-intervals

$$(\mathrm{D}^2+4\beta^2)\, w = 2\mathrm{P}|\lambda|\cosh 2\alpha(x-a), \qquad (\mathrm{D}^3+4\beta^2\mathrm{D})\, w = 2\mathrm{Q}\alpha|\lambda|\sinh 2\alpha(x-a)$$

where P, Q are quantities lying between $\Pi_1\tau^2$ and $\Pi_2\tau^2$, if we use Π_1 to denote the product of all the factors > 1 and Π_2 that of all the factors < 1 among the quantities τ^2.

If $\log r$ is of uniformly limited total fluctuation then $\Pi_1\tau^2$ and $\Pi_2\tau^2$ are limited, and so, therefore, are P, Q.

9. Now

$$\begin{aligned} |\lambda v^2| + |Dv|^2 &= (\alpha^2+\beta^2)\, v\bar{v} + Dv D\bar{v}, \\ &= \tfrac{1}{2}(D^2+4\beta^2)\, w, \\ &= P|\lambda| \cosh 2\alpha\,(x-a), \end{aligned}$$

and hence $|\lambda v^2|$ and $|Dv|^2$ are severally less than this.

Also, the real part of $\sqrt{-\lambda} v D\bar{v}$ is

$$\begin{aligned} \tfrac{1}{2}\{(\alpha+\iota\beta)\, vD\bar{v} + (\alpha-\iota\beta)\, \bar{v}Dv\} &= \frac{1}{8\alpha} D(D^2+4\beta^2)\, w, \\ &= \tfrac{1}{4}Q|\lambda| \sinh 2\alpha\,(x-a). \quad . \quad . \quad . \quad . \quad . \quad . \quad (7) \end{aligned}$$

Hence $|v|$ and $|Dv \div \sqrt{\lambda}|$ are both

$$< \{P \cosh 2\alpha\,(x-a)\}^{1/2},$$

but

$$> \tfrac{1}{4}Q \sinh 2\alpha\,(x-a)\,\{P \cosh 2\alpha\,(x-a)\}^{-1/2}.$$

Now P, Q cannot increase or decrease indefinitely, and therefore, if we do not allow α to tend to zero, we have that $|v|$ and $|Dv \div \sqrt{\lambda}|$ bear to $\exp \alpha\,(x-a)$ ratios which are limited in both directions; and this is true, both when the sub-intervals are increased in number and also when $|\lambda|$ is increased indefinitely.

In the same way it may be seen that the ratios of $|u\sqrt{\lambda}|$ and $|Du|$ to $\exp \alpha\,(x-a)$ are limited in both directions.

When $\alpha = 0$ the argument shows that $|u\sqrt{\lambda}|$, $|Du|$, $|v|$, and $|Dv \div \sqrt{\lambda}|$ are limited above, but not below, and, in fact, we know that each of the four is capable of vanishing.

10. The condition that $\log r$ should be of uniformly limited total fluctuation is necessary, for if it is not fulfilled P may be indefinitely great or small, and it is conceivable that $u\sqrt{\lambda}$ and Du, for instance, should become very small together, in the same way that a pendulum would be practically stopped if its velocity were suddenly reduced in a constant ratio at every passage through the lowest position and increased in the same ratio at every time of reaching one of the extreme positions, when, of course, the increase would be of no effect.

11. Hence if in (5) we take the limits to be a, b and put $\rho' = r$, a local average of ρ, $\sigma' = 0$, $\lambda' = \lambda$, $\phi = \phi(x, a)$, $\phi' = u(x, b)$, we have

$$\begin{aligned} \phi(b, a) + u(a, b) &= \phi(b, a) - u(b, a) \\ &= \int_a^b \left\{(\rho - r)\,\Phi(x, a)\,U(x, b) + \lambda\left(\frac{1}{\rho} - \frac{1}{r}\right)\phi(x, a)\,u(x, b) - \sigma\phi(x, a)\,u(x, b)\right\} dx. \end{aligned}$$

Changing ϕ into ψ, or u into v, or both, we have similarly

$$-\Phi(b, a)+v(a, b) = -\Phi(b, a)+\mathrm{U}(b, a)$$
$$= \int_a^b \left\{(\rho-r)\,\Phi(x, a)\,\mathrm{V}(x, b)+\lambda\left(\frac{1}{\rho}-\frac{1}{r}\right)\phi(x, a)\,v(x, b)-\sigma\phi(x, a)\,v(x, b)\right\} dx,$$

$$\psi(b, a)-v(b, a)$$
$$= \int_a^b \left\{(\rho-r)\,\Psi(x, a)\,\mathrm{U}(x, b)+\lambda\left(\frac{1}{\rho}-\frac{1}{r}\right)\psi(x, a)\,u(x, b)-\sigma\psi(x, a)\,u(x, b)\right\} dx,$$

$$-\Psi(b, a)+\mathrm{V}(b, a)$$
$$= \int_a^b \left\{(\rho-r)\,\Psi(x, a)\,\mathrm{V}(x, b)+\lambda\left(\frac{1}{\rho}-\frac{1}{r}\right)\psi(x, a)\,v(x, b)-\sigma\psi(x, a)\,v(x, b)\right\} dx.$$

These are expressions for the errors committed when u, v, U, V are taken as the values of ϕ, ψ, Φ, Ψ.

Let μ denote the upper boundary of the ratios of

$$|\sqrt{\lambda}\,\{\phi(x_1, x_0)-u(x_1, x_0)\}|, \qquad |\Phi(x_1, x_0)-\mathrm{U}(x_1, x_0)|,$$
$$|\psi(x_1, x_0)-v(x_1, x_0)|, \qquad |\lambda^{-1/2}\{\Psi(x_1, x_0)-\mathrm{V}(x_1, x_0)\}|$$

to $\exp \alpha(x_1-x_0)$ for values of x_0, x_1 such that

$$a \leq x_0 < x_1 \leq b.$$

Let the symbol // denote equality in order of magnitude so that P//Q means that the ratios P/Q, Q/P are both limited.

Then in the expressions for the four errors, since $b \geq x \geq a$,

$\lambda^{1/2}u(x, b)$, $\quad v(x, b)$, $\quad \mathrm{U}(x, b)$, $\quad \lambda^{-1/2}\mathrm{V}(x, b)$ are all at most $//\exp\alpha(b-x)$,

while

$\lambda^{1/2}\phi(x, a)$, $\quad \psi(x, a)$, $\quad \Phi(x, a)$, $\quad \lambda^{-1/2}\Psi(x, a)$ are all at most $//(1+\mu)\exp\alpha(x-a)$.

Hence such a product as $\Phi(x, a)\,\mathrm{U}(x, b)$ or $\lambda\phi(x, a)\,u(x, b)$ is $//(1+\mu)\exp\alpha(b-a)$ at most, and

$$\left|\int_a^b \left\{(\rho-r)\,\Phi(x, a)\,\mathrm{U}(x, b)+\lambda\left(\frac{1}{\rho}-\frac{1}{r}\right)\phi(x, a)\,u(x, b)\right\} dx\right|$$

is at most $//\varpi(1+\mu)\exp\alpha(b-a)$ where

$$\varpi = \int_a^b \left\{|\rho-r| + \left|\frac{1}{\rho}-\frac{1}{r}\right|\right\} dx.$$

Also $\int_a^b |\sigma|\,dx$ is supposed to be finite, and therefore

$$\int_a^b \sigma\phi(x, a)\,u(x, b)\,dx \text{ is at most } // \frac{1+\mu}{\lambda}\exp\alpha(b-a)$$

Thus the error in $\lambda^{1/2}\phi(b, a)$ is at most $//$

$$(1+\mu)\{\varpi|\lambda|^{1/2}+|\lambda|^{-1/2}\}\exp\alpha(b-a),$$

and the same may be proved for the errors in

$$\psi(b, a), \qquad \Phi(b, a), \qquad \lambda^{1/2}\Psi(b, a);$$

also we may put x_1, x_0 for b, a. Hence, at most

$$\mu // (1+\mu)\{\varpi|\lambda|^{1/2}+|\lambda|^{-1/2}\},$$

and ϖ may be as small as we please. If we make $\varpi // \lambda^{-1}$ we have

$$\mu // |\lambda|^{-1/2},$$

so that in each of the four cases the error is of this order relatively to the true value.

12. Since u, v, U, V do not depend on σ, the error produced by neglecting or altering σ is also of the order of $\lambda^{-1/2}$ in comparison with the true value.

From (5) by putting $\rho=\rho'$, $\sigma=\sigma'$, and making λ' approach λ, we can deduce such results as

$$\frac{d}{d\lambda}\phi(b, a)=\int_a^b \frac{1}{\rho}\phi(x, a)\phi(x, b)\,dx, \qquad \frac{d}{d\lambda}\Psi(b, a)=-\int_a^b \frac{1}{\rho}\psi(x, a)\psi(x, b)\,dx, \quad (8)$$

thus proving that ϕ, ψ, Φ, Ψ, have everywhere finite differential coefficients with respect to the complex variable λ. They are therefore holomorphic functions of λ all over the plane.

13. It is sometimes useful to know that Φ and ψ cannot tend to destroy each other in such an expression as $\Phi+k\psi$, where k is positive.

To see this, begin at the other end of the interval (a, b). The argument of §§ 8, 9 proves that the real part of $\sqrt{-\lambda}v(x, b)\,\mathrm{D}\bar{v}(x, b)$ is $-\frac{1}{4}|\lambda|\,\mathrm{Q}_1\sinh 2\alpha(b-x)$ where Q_1 is positive and limited both ways. In this put $x=a$, and subtract from a multiple of the result of putting b for x in (7).

Since $v(a, b)=\mathrm{U}(b, a)$ and $\mathrm{V}(a, b)=-\mathrm{V}(b, a)$ the real part of

$$\sqrt{-\lambda}\{\mathrm{U}(b, a)+kv(b, a)\}\,\overline{\mathrm{V}}(b, a)$$

is thus found to be

$$\tfrac{1}{4}|\lambda|\left\{\frac{\mathrm{Q}_1}{r_0}+\frac{k\mathrm{Q}}{r_1}\right\}\sinh 2\alpha(b-a),$$

where r_0, r_1 are the values of r at a, b respectively.

Since k, Q, Q_1, r_0, r_1 are positive, and Q, Q_1, r_0, r_1 are limited in both directions, the ratio of $\mathrm{U}(b,a)+kv(b,a)$ to $\exp\alpha(b-a)$ is also limited in both directions. Combining this with the results of § 11 we have the following theorem:—*The values of* $\sqrt{\lambda}\,\phi(b, a)$, $\psi(b, a)$, $\Phi(b, a)$, $\Psi(b, a)/\sqrt{\lambda}$ *and also* $\Phi(b, a)+k\psi(b, a)$ *where* k *is a* *positive constant, are all of the same order of magnitude as* $\exp \alpha (b-a)$, *and cannot be of lower order unless* α *tends to zero.*

14. The values of $u(b, a)$, $U(b, a)$ ... can be written down as follows:—

Let the successive intervals into which $b-a$ is divided be denoted by $\theta_1, \theta_2, \ldots, \theta_n$, and the values of r in those intervals respectively by $r_1, r_2, \ldots, r_n$; also let $\lambda = -l^2$. Then

$$u(b, a) = \frac{1}{2^n l} \Sigma \frac{(\epsilon_1 r_1 + \epsilon_2 r_2)(\epsilon_2 r_2 + \epsilon_3 r_3) \ldots (\epsilon_{n-1} r_{n-1} + \epsilon_n r_n)}{\epsilon_2 r_2 \cdot \epsilon_3 r_3 \ldots \epsilon_{n-1} r_{n-1}} \exp l(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots + \epsilon_n\theta_n)$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are all ± 1 and Σ refers to the 2^n ways of taking them. The product in the numerator of the fractional factor contains $n-1$ binomial factors.

$$U(b, a) = \frac{1}{2^n} \Sigma \frac{(\epsilon_1 r_1 + \epsilon_2 r_2) \ldots (\epsilon_{n-1} r_{n-1} + \epsilon_n r_n)}{\epsilon_2 r_2 \ldots \epsilon_n r_n} \exp l(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots + \epsilon_n\theta_n),$$

$$v(b, a) = \frac{1}{2^n} \Sigma \frac{(\epsilon_1 r_1 + \epsilon_2 r_2) \ldots (\epsilon_{n-1} r_{n-1} + \epsilon_n r_n)}{\epsilon_1 r_1 \ldots \epsilon_{n-1} r_{n-1}} \exp l(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots + \epsilon_n\theta_n),$$

$$V(b, a) = \frac{l}{2^n} \Sigma \frac{(\epsilon_1 r_1 + \epsilon_2 r_2) \ldots (\epsilon_{n-1} r_{n-1} + \epsilon_n r_n)}{\epsilon_1 r_1 \ldots \epsilon_n r_n} \exp l(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots + \epsilon_n\theta_n).$$

It may, in fact, be verified that

$$\frac{du}{d\theta_n} = r_n U, \qquad \frac{dU}{d\theta_n} = \frac{l^2}{r_n} u,$$

so that u, U satisfy the differential equations assigned; also, when $\theta_n = 0$ the expressions reduce to the corresponding ones for $n-1$ intervals, and therefore u, U are continuous throughout as functions of b; lastly, at the beginning of the second interval

$$u = \frac{r_1}{l} \sinh l\theta_1, \qquad U = \cosh l\theta_1.$$

This completes the verification for u, U, and V, v may be treated similarly.

The approximations generally used in the treatment of the equation (1) might be derived from these by neglecting all the terms except those in which $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_n$, that is, the terms which contain one or more of the differences $r_1 - r_2$, $r_2 - r_3$, ... as factors, and in the two terms which are left putting $2\sqrt{r_m r_{m+1}}$ for $r_m + r_{m+1}$. Thus $u(b, a)$, for instance, would reduce to

$$\sqrt{r_1 r_n} \sinh l(b-a).$$

This simplified form is clearly not admissible unless all the differences $r_1 - r_2$, $r_2 - r_3$, ... are small.

15. Now suppose ρ to be limited both ways, and let $F(t, x, R)$ denote

$$\frac{1}{\rho_t}\int_{(R)} \frac{\psi(x, 0)\,\psi(t, 1)}{\Psi(1, 0)}\, d\lambda,$$

where ρ_t means the value of the function ρ of the argument t, and the path of integration in the λ-plane indicated by (R) is a circle of radius R with centre at the origin, R being such that this path does not pass through a point where $\Psi(1, 0) = 0$.

$\rho_t F(t, x, R)$ is a symmetric function of t, x for

$$\rho_t F(t, x, R) - \rho_x F(x, t, R) = \int_{(R)} \{\Psi(1, 0)\}^{-1} \{\psi(x, 0)\,\psi(t, 1) - \psi(t, 0)\,\psi(x, 1)\}\, d\lambda$$
$$= \int_{(R)} \phi(x, t)\, d\lambda = 0,$$

since $\phi(x, t)$ is a holomorphic function of λ everywhere (§ 12).

It will now be proved that $F(t, x, R)$ satisfies the conditions of HOBSON's convergence theorem ('Proc. L.M.S.,' ser. 2, vol. 6, pp. 350–1), that is—

(1) Its absolute value does not exceed a certain quantity $\overline{F}$ for all values of t, x such that $t \sim x \geqq \mu$ and for all values of R.

(2) $\int_a^b F(t, x, R)\, dt$ exists for all values of a, b such that $0 \leqq a < b \leqq 1$ and for each value of x in the interval (0, 1) which does not lie between $a - \mu$ and $b + \mu$; this integral, moreover, is less than a positive number A, independent of a, b, x.

(3) $A \rightarrow 0$ when $R \rightarrow \infty$.

16. A change in the value of R does not affect $F(t, x, R)$ unless it changes the number of zeros of $\Psi(1, 0)$ enclosed by the circle: hence we may suppose the circle to cross the real axis on the positive side at a point T, where $v(1, 0)$ is zero. Thus, at the point T, $V(1, 0)$, and therefore also $\Psi(1, 0)$, are $//\sqrt{\lambda}$, since $|v|^2 + |Dv|^2/|\lambda|$ cannot tend to zero.

Again (§ 12)

$$\frac{d}{d\lambda}\Psi(1, 0) = -\int_0^1 \frac{1}{\rho}\psi(x, 0)\,\psi(x, 1)\, dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is limited when α is limited,* whatever the value of β. A distance $//\sqrt{\lambda}$ can therefore be assigned such that within that distance of T $|\Psi(1, 0) \div \sqrt{\lambda}|$ does not approach zero, but exceeds a certain fixed quantity independent of R. Beyond that distance from T on the path it has been proved already (§ 11) that $\Psi(1, 0)\ //\lambda^{1/2} \exp \alpha$, so that this is now proved for the whole path.

The numerator $\psi(x, 0)\,\psi(t, 1)\,(t > x)$ is $//\exp \alpha x \,.\, \exp \alpha(1-t)$, so that the subject of integration in $F(t, x, R)$ is

$$//\lambda^{-1/2} \exp \alpha(x-t), \quad \text{that is,} \quad \lambda^{-1/2} \exp(-\alpha\mu) \text{ at most.}$$

* We still take $\sqrt{-\lambda} = \alpha + \iota\beta$; thus α is limited for points within a distance $//\sqrt{\lambda}$ (or $\sqrt{R}$) of T.

Also $\lambda^{-1/2}\, d\lambda \,//\, d\alpha$, except where α is near its maximum, and in that part of the path the factor $\exp(-\alpha\mu)$ is so small that the contribution to the integral is negligible.

Hence $\mathrm{F}(t, x, \mathrm{R})$ is at most $// \int_0^\infty \exp(-\alpha\mu)\, d\alpha$, that is, $\frac{1}{\mu}$.

This is the first of HOBSON'S conditions.

Again

$$\int_a^b \mathrm{F}(t, x, \mathrm{R})\, dt = \int_{(\mathrm{R})} \frac{\psi(x, 0)}{\lambda\Psi(1, 0)} \{\Psi(a, 1) - \Psi(b, 1)\}\, d\lambda + \int_a^b \int_{(\mathrm{R})} \frac{\sigma}{\lambda} \frac{\psi(x, 0)\,\psi(t, 1)}{\Psi(1, 0)}\, d\lambda\, dt. \quad . \quad (9)$$

In the first term of this expression the first term of the subject of integration contains the factor

$$\psi(x, 0)\,\Psi(a, 1) \div \Psi(1, 0),$$

which is of the order of $\exp \alpha(x-a)$, that is, at most $\exp(-\alpha\mu)$ when we take $x < a - \mu$. Also $\lambda^{-1/2}\, d\lambda \,//\, d\alpha$ as before and $\lambda^{-1/2} \,//\, \mathrm{R}^{-1/2}$. Hence the contribution of this term is $// \frac{1}{\mu\sqrt{\mathrm{R}}}$ at most, and the same is true of the second part of the first term.

In the other term the integral of $|\sigma|$ is finite, and the integration with respect to t is over a finite range, so that these two elements do not affect the order of magnitude: the factor $\psi(x, 0)\,\psi(t, 1) \div \lambda\Psi(1, 0)$ is of the order of $\lambda^{-3/2} \exp \alpha(x-t)$, that is, $\mathrm{R}^{-3/2}$ at most, even when $\mu = 0$: the length of path in the λ-plane is $2\pi\mathrm{R}$. Hence the contribution of this term is $// \mathrm{R}^{-1/2}$ independently of μ if $x \leq a$.

On account of the symmetry between x and t, like results can be deduced if $x > b + \mu$.

Thus the second and third of HOBSON'S conditions are fulfilled.

17. Again, so long as $x \leq a$ the value of $\psi(x, 0)\,\Psi(a, 1) \div \Psi(1, 0)$ is limited, and so is that of $\psi(x, 0)\,\Psi(b, 1) \div \Psi(1, 0)$. Hence the first term on the right in (9) is limited, the integral of $\left|\frac{d\lambda}{\lambda}\right|$ being 2π. The second term has been found to tend to zero when R is increased, and therefore

$$\int_a^b \mathrm{F}(t, x, \mathrm{R})\, dt$$

is limited if $x \leq a < b$, or similarly if $x \geq b > a$. When $a < x < b$ we may write

$$\int_a^b = \int_a^x + \int_x^b,$$

so that $\int_a^b \mathrm{F}(t, x, \mathrm{R})\, dt$ is the sum of two terms, each limited, and is itself limited for all values of a, b, x, R. This covers one of HOBSON'S further conditions (§ 4 of his paper, p. 361).

It is not clear that the two integrals

$$\int_x^{x\pm\mu} \mathrm{F}(t, x, \mathrm{R})\, dt$$

tend to definite limits when $\mathrm{R} \to \infty$, but their difference does so, and, in fact, if $a < x < b$,

$$\int_a^b \mathrm{F}(t, x, \mathrm{R})\, dt = \int_a^x + \int_x^b \text{ of the same}$$

$$= \int_{(\mathrm{R})} \left[\frac{\psi(x, 1)}{\lambda\Psi(1, 0)} \{\Psi(a, 0) - \Psi(x, 0)\} + \frac{\psi(x, 0)}{\lambda\Psi(1, 0)} \{\Psi(x, 1) - \Psi(b, 1)\} \right] d\lambda$$

$$+ \int_a^x \int_{(\mathrm{R})} \frac{\sigma}{\lambda} \frac{\psi(x, 1)\, \psi(t, 0)}{\Psi(1, 0)}\, d\lambda\, dt + \int_x^b \int_{(\mathrm{R})} \frac{\sigma}{\lambda} \frac{\psi(x, 0)\, \psi(t, 1)}{\Psi(1, 0)}\, d\lambda\, dt. \quad . \quad . \quad (10)$$

All the parts of this expression tend to zero when $\mathrm{R} \to \infty$ except

$$\int_{(\mathrm{R})} \frac{-\psi(x, 1)\, \Psi(x, 0) + \Psi(x, 1)\, \psi(x, 0)}{\lambda\Psi(1, 0)}\, d\lambda,$$

which

$$= -\int_{(\mathrm{R})} \frac{d\lambda}{\lambda} = -2\iota\pi.$$

18. It is now possible to prove that *if $f(x)$ is continuous at x and of limited total fluctuation in a neighbourhood of x then*

$$f(x) = -\lim_{\mathrm{R}\to\infty} \frac{1}{2\iota\pi} \int_0^1 f(t)\, \mathrm{F}(t, x, \mathrm{R})\, dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

For (1) this holds when $f(x)$ has a constant value (§ 17);

(2) The contribution to the integral from values of t not lying between $x \pm \mu$ may be ignored, the values $x \pm \mu$ lying within the neighbourhood where $f(x)$ is of limited total fluctuation (HOBSON's convergence theorem);

(3) By the second mean-value theorem, if $f_1(x)$ is monotone,

$$\int_x^{x+\mu} \{f_1(t) - f_1(x)\}\, \mathrm{F}(t, x, \mathrm{R})\, dt = \{f_1(x+\mu) - f_1(x)\} \int_{x+\mu_1}^{x+\mu} \mathrm{F}(t, x, \mathrm{R})\, dt,$$

where $0 \leqq \mu_1 < \mu$. In the last expression the second factor is finite (§ 17), while the first can be made as small as we please by taking μ small enough, if f_1 is supposed continuous. The same holds for the integral from $x-\mu$ to x.

Thus if $f(x)$ is of limited total fluctuation between $x \pm \mu$ and is, in fact, the sum of two functions f_1, f_2 which are monotone between those limits, and continuous at x,

$$-2\iota\pi f(x) = f(x) \times \lim_{\mathrm{R}\to\infty} \int_0^1 \mathrm{F}(t, x, \mathrm{R})\, dt$$

$$= \lim_{\mathrm{R}\to\infty} \left[\int_0^1 f(t)\, \mathrm{F}(t, x, \mathrm{R})\, dt + \left\{ \int_0^{x-\mu} + \int_{x-\mu}^x + \int_x^{x+\mu} + \int_{x+\mu}^1 \right\} \{f_1 x - f_1 t\}\, \mathrm{F}(t, x, \mathrm{R})\, dt \right.$$

$$\left. + \left\{ \int_0^{x-\mu} + \int_{x-\mu}^x + \int_x^{x+\mu} + \int_{x+\mu}^1 \right\} \{f_2 x - f_2 t\}\, \mathrm{F}(t, x, \mathrm{R})\, dt \right]$$

where the last eight integrals all tend to zero when R is increased without limit, and, therefore,

$$f(x) = -\frac{1}{2\iota\pi} \operatorname*{Lim}_{R\to\infty} \int_0^1 f(t)\, F(t, x, R)\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which was to be proved.

From this result one of the expansions of STURM and LIOUVILLE can be deduced by considering the singularities of the subject of integration in $F(t, x, R)$, that is, the values of λ for which $\Psi(1, 0) = 0$.

When $\Psi(1, 0) = 0$ we have $\psi(x, 0)$ and $\psi(x, 1)$ the same but for a constant factor, since

$$\Phi(1, 0)\,\psi(x, 0) - \Psi(1, 0)\,\phi(x, 0) = \psi(x, 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Also (§ 12),

$$\frac{d}{d\lambda}\Psi(1, 0) = -\int_0^1 \frac{1}{\rho}\psi(x, 0)\,\psi(x, 1)\,dx.$$

Hence, the residue of $F(t, x, R)$ is

$$-\frac{1}{\rho_t}\psi(x, 0)\,\psi(t, 0) \div \int_0^1 \frac{1}{\rho}\{\psi(x, 0)\}^2\,dx,$$

and we have

$$f(x) = \Sigma\psi(x, 0)\int_0^1 \frac{1}{\rho}\psi(x, 0) f(x)\,dx \div \int_0^1 \frac{1}{\rho}\{\psi(x, 0)\}^2\,dx \quad . \quad . \quad . \quad (13)$$

the summation referring to the infinite series of values of λ *for which* $\Psi(1, 0) = 0$, *taken in ascending order of magnitude; thus* $f(x)$ *is expanded in a series of functions* ϕ *satisfying* (3) *and such that* Φ *vanishes at each of the extreme values* 0, 1.

In order to investigate the validity of the expansion when $x = 1$ we need to discuss

$$\int_a^1 F(t, 1, R)\,dt$$

which

$$= \int_{(R)} \frac{\Psi(a, 0) - \Psi(1, 0)}{\lambda\Psi(1, 0)}\,d\lambda + \int_a^1 \int_{(R)} \frac{\sigma}{\lambda}\,\frac{\psi(t, 0)}{\Psi(1, 0)}\,d\lambda\,dt.$$

Here the only term of importance is the second part of the first integral, which has the value $-2\iota\pi$. From this it follows, in the same way, that the expansion holds good at the upper limit, and a like result can be proved when $x = 0$; in each case it is supposed that $f(x)$ is continuous and of limited total fluctuation in the neighbourhood.

The course of the proof, moreover, shows that the series is *uniformly* convergent so long as x lies within an interval which is contained within another interval in which $f(x)$ is continuous and of limited total fluctuation.

It has been supposed that ρ is limited both ways. When this is not so, but the integrals of ρ and $\frac{1}{\rho}$ exist, the argument still applies if we detach the factor $\frac{1}{\rho_t}$ from

$F(t, x, R)$ and group it with dt in the integrations, that is, if we think of $\int^t \frac{dx}{\rho}$ as the variable of integration in such expressions as

$$\int f(t)\, F(t, x, R)\, dt \quad \text{and} \quad \int F(t, x, R)\, dt.$$

19. The same method can be applied when $F(t, x, R)$ has the more general value

$$\frac{1}{\rho_t}\int_{(R)} \frac{\omega(t, x)}{\Omega}\, d\lambda,$$

where

$$\omega(t, x) = K\psi(x, 0)\psi(t, 1) + H\phi(x, 0)\psi(t, 1) - G\psi(x, 0)\phi(t, 1) - E\phi(x, 0)\phi(t, 1) - L\phi(x, t),$$

$$\Omega = K\Psi(1, 0) + H\Phi(1, 0) + G\psi(1, 0) + E\phi(1, 0) - 2L,$$

E, G, H, K, L are real constants, with the one proviso that when K is 0, GH is positive or zero, so that the terms in Ω involving G, H cannot tend to destroy each other (§ 13). Ω cannot vanish except for real values of λ if $GH - EK - L^2$ is zero or positive,* a condition which includes the proviso made.

Thus it still follows that when $0 < x < 1$

$$f(x) = -\lim_{R\to\infty} \frac{1}{2\iota\pi}\int_0^1 f(t)\, F(t, x, R)\, dt$$

$$= \Sigma \int_0^1 \frac{1}{\rho_t} f(t)\, \omega(t, x)\, dt \Big/ \int_0^1 \frac{1}{\rho}\, \omega(x, x)\, dx, \quad \ldots \ldots \quad (14)$$

the summation referring to all the values of λ *for which*

$$\Omega = 0, \quad \ldots \ldots \ldots \ldots \quad (15)$$

these values being taken in ascending order of magnitude.

Sufficient conditions for the validity of this expansion are those already stated, namely, that $f(x)$ *should be continuous at* x *and of limited total fluctuation within some neighbourhood containing* x *as an internal point.*

The *restrictions placed on the functions* σ, ρ in the fundamental equations (3) are that—

(i) ρ *shall be positive;*

(ii) *The integrals of* $|\sigma|$, ρ, $\frac{1}{\rho}$ *shall exist;*

(iii) *A local average of* ρ *shall have a logarithm of uniformly limited total fluctuation* for some method of division into sub-intervals of the fundamental domain

* For this and other results see 'Proc. L.M.S.,' ser. 2, vol. 3, pp. 86–90, and vol. 5, p. 420; the former passage contains a discussion of the case of equal roots.

(0, 1). Here the term "local average" means a function r constant in each sub-interval, and such that

$$\int_0^1 |\rho - r|\, dx \quad \text{and} \quad \int_0^1 \left|\frac{1}{\rho} - \frac{1}{r}\right| dx$$

tend to zero when the greatest of the sub-intervals does so. If $\frac{1}{\rho}$ is limited, r may be the actual average of ρ in each sub-interval.

The expansions discussed by LIOUVILLE and STURM are those for which in the present notation

$$\mathrm{L} = 0, \quad \mathrm{GH} = \mathrm{EK}.$$

Thus

$$\mathrm{K}\omega(t, x) = \{\mathrm{K}\psi(x, 0) + \mathrm{H}\phi(x, 0)\}\{\mathrm{K}\psi(t, 1) - \mathrm{G}\phi(t, 1)\},$$

and since

$$\omega(t, x) - \omega(x, t) = \Omega\phi(x, t)$$
$$= 0 \quad \text{when} \quad \Omega = 0,$$

the typical term may be written as a multiple of

$$\mathrm{K}\psi(x, 0) + \mathrm{H}\phi(x, 0) \quad \text{or} \quad \mathrm{K}\psi(x, 1) - \mathrm{G}\phi(x, 1)$$

indifferently, that is, it satisfies the equations (3) and is such that

$$\text{when } x = 0, \quad \mathrm{K}\Phi = \mathrm{H}\phi,$$

and

$$\text{when } x = 1, \quad \mathrm{K}\Phi = -\mathrm{G}\phi.$$

20. It may also be proved that if $f(x)$ is continuous and is equal to the sum of the former series (§ 18), the new expansion holds also. For if $\mathrm{F}_1(t, x, \mathrm{R})$ is the special function $\mathrm{F}(t, x, \mathrm{R})$ in which H, G, E, L are zero, that is, the function denoted by $\mathrm{F}(t, x, \mathrm{R})$ in § 15, we have, after some reduction,

$$\rho_t\{\mathrm{F}_1(t, x, \mathrm{R}) - \mathrm{F}(t, x, \mathrm{R})\}$$
$$= \int_{(\mathrm{R})} \frac{1}{\Omega\Psi(1, 0)} \Big[\mathrm{H}\psi(x, 1)\psi(t, 1) + \mathrm{G}\psi(x, 0)\psi(t, 0) + \mathrm{E}\{\phi(x, 0)\psi(t, 0) - \phi(x, 1)\psi(t, 1)\}$$
$$+ \mathrm{L}\{\phi(x, t)\Psi(1, 0) - 2\psi(x, 0)\psi(t, 1)\}\Big] d\lambda, \quad \ldots \ldots \quad (16)$$

which, even in the unfavourable case when K = 0, is finite when $\mathrm{R} \to \infty$, unless x, t are each equal to one of the limiting values 0, 1.

Now the difference of the two expansions for $f(x)$ is

$$\operatorname*{Lim}_{\mathrm{R}\to\infty} \frac{1}{2\iota\pi}\int_0^1 f(t)\{\mathrm{F}(t, x, \mathrm{R}) - \mathrm{F}_1(t, x, \mathrm{R})\}\, dt,$$

that is,

$$\operatorname*{Lim}_{\mathrm{R}\to\infty} \frac{1}{2\iota\pi}\int_0^1 \{f(t) - f(x)\}\{\mathrm{F}(t, x, \mathrm{R}) - \mathrm{F}_1(t, x, \mathrm{R})\}\, dt,$$

for when the function to be expanded is constant, each of the expansions holds.

In this expression the parts depending on F, F_1 are separately negligible by HOBSON'S theorem, except for values of t between $x \pm \mu$, and for such values $f(t) - f(x)$ is small and $F - F_1$ is finite. The whole is therefore arbitrarily small and, in fact, zero; thus if the one expansion is valid for $f(x)$, so is the other,* provided that $f(x)$ is continuous at x.

21. Again, the error produced in one of the functions ϕ, ψ, Φ, Ψ by neglecting or altering σ is relatively of the order of $\lambda^{-1/2}$, and thus it follows that if F, F_1 are functions of the present type (§ 19) the same except for a change in σ then $F(t, x, R) - F_1(t, x, R)$ is limited. Similarly, then, the expansibility of $f(x)$ is not affected by a change in the value of σ in the fundamental integral equations (3) so long as $\int |\sigma| \, dx$ exists and $f(x)$ is continuous at x.

22. In order that the more general expansion may hold at the limits 0, 1, some further conditions are necessary. It does not seem worth while to discuss these in detail, but they are satisfied—

(1) When $f(1) = f(0) = 0$; and

(2) When $f(1) = f(0)$ and $G = H = L = 1$, $K = E = 0$, this being the case of a periodic function expanded in a series of periodic terms, since

$$\Phi(1, 0) + \psi(1, 0) = 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

is the condition that functions ϕ, Φ may satisfy the equations (3) and have the same values at both ends of the interval (0, 1).

On account of the periodicity there is no occasion to distinguish between the two end-points or between these and the other points of the domain.

23. It is known that the Fourier constants a_n, b_n of a function $f(x)$ are such as to give the least possible value to

$$\int_{-\pi}^{\pi} \{f(x) - \Sigma a_n \cos nx - \Sigma b_n \sin nx\}^2 \, dx$$

and there is a similar theorem for the present expansions.

The condition that there may be functions $\xi(x)$, $\Xi(x)$ satisfying (3) and fulfilling the boundary conditions

$$K\Xi(1) + G\xi(1) = \theta\xi(0), \qquad H\Xi(1) + E\xi(1) = \theta\Xi(0) \quad . \quad . \quad . \quad . \quad (18)$$

is

$$\begin{vmatrix} K\Psi(1, 0) + G\psi(1, 0) - \theta, & H\Psi(1, 0) + E\psi(1, 0) \\ K\Phi(1, 0) + G\phi(1, 0), & H\Phi(1, 0) + E\phi(1, 0) - \theta \end{vmatrix} = 0,$$

* This result and that of § 21 were first given for the usual LIOUVILLE series by J. MERCER (see 'Roy. Soc. Proc.,' A, vol. 84, pp. 573–5, and 'Phil. Trans.,' A, vol. 211, p. 147).

which reduces to

$$\Omega = 0 \,. \quad \dots \dots \dots \dots \quad (15)$$

if θ is a root of the equation

$$\theta^2 - 2\mathrm{L}\theta + (\mathrm{GH} - \mathrm{EK}) = 0. \quad \dots \dots \dots \quad (19)$$

When $\mathrm{GH} - \mathrm{EK} - \mathrm{L}^2$ is positive, θ_1, θ_2, the two roots of (19), are conjugate complex quantities, and they may still be considered so when $\mathrm{GH} - \mathrm{EK} = \mathrm{L}^2$ and θ_1, θ_2 are real and equal. Let ξ, η be the two corresponding solutions of the fundamental equations (3); these are also conjugate. Let coefficients

$$a_1,\ a_2,\ \dots,\ a_n,\ \dots,\ a_m$$

and

$$b_1,\ b_2,\ \dots,\ b_n,\ \dots,\ b_m$$

be so determined that a_n, b_n are conjugate for all values of n, and that

$$\int_0^1 \frac{1}{\rho} \{f(x) - a_1\xi_1(x) - \dots - a_m\xi_m(x)\} \{f(x) - b_1\eta_1(x) - \dots - b_m\eta_m(x)\} \, dx \quad . \quad (20)$$

is the least possible. Here the different values of λ satisfying the equation (15) and the corresponding values of ξ, η are distinguished by suffixes. Thus the two factors in the subject of integration are conjugate imaginaries and the integral is essentially positive.

From (5), § 4, we have

$$(\lambda_1 - \lambda_2) \int_0^1 \frac{1}{\rho} \xi_1(x)\, \eta_2(x)\, dx = \begin{vmatrix} \xi_1(1) & \Xi_1(1) \\ \eta_2(1) & \mathrm{H}_2(1) \end{vmatrix} - \begin{vmatrix} \xi_1(0) & \Xi_1(0) \\ \eta_2(0) & \mathrm{H}_2(0) \end{vmatrix}$$

$$= \begin{vmatrix} \xi_1(1) & \Xi_1(1) \\ \eta_2(1) & \mathrm{H}_2(1) \end{vmatrix} \left\{1 - \frac{\mathrm{GH} - \mathrm{EK}}{\theta_1\theta_2}\right\} \text{ from (18)}$$

$$= 0, \text{ since } \theta_1\theta_2 = \mathrm{GH} - \mathrm{EK}.$$

Similarly,

$$\int_0^1 \frac{1}{\rho} \xi_m(x)\, \eta_n(x)\, dx = 0$$

for any unequal suffixes m, n.

Hence it readily follows that (20) is a minimum when

$$a_n = \int_0^1 \frac{1}{\rho} f(x)\, \eta_n(x)\, dx \div \int_0^1 \frac{1}{\rho} \xi_n(x)\, \eta_n(x)\, dx,$$

$$b_n = \int_0^1 \frac{1}{\rho} f(x)\, \xi_n(x)\, dx \div \int_0^1 \frac{1}{\rho} \xi_n(x)\, \eta_n(x)\, dx,$$

while the general term in the expansion of $f(x)$ (§ 19) is

$$\tfrac{1}{2} \{a_n\xi_n(x) + b_n\eta_n(x)\}$$

with these values of a_n, b_n, since it may be verified that when $\lambda = \lambda_n$

$$\xi(t)\,\eta(x)\,\{\mathrm{K}\psi(1,0)+\mathrm{H}\phi(1,0)\} = \xi(1)\,\eta(1)\,\{\omega(t,x)+(\mathrm{L}-\theta_1)\,\phi(x,t)\},$$

$$\xi(x)\,\eta(t)\,\{\mathrm{K}\psi(1,0)+\mathrm{H}\phi(1,0)\} = \xi(1)\,\eta(1)\,\{\omega(t,x)+(\mathrm{L}-\theta_2)\,\phi(x,t)\},$$

and therefore, by addition, $\omega(t,x)$ is the same as

$$\xi(t)\,\eta(x)+\xi(x)\,\eta(t)$$

but for a constant factor, since $\theta_1+\theta_2 = 2\mathrm{L}$. (Compare 'Proc. L.M.S.,' ser. 2, vol. 5, p. 473.)

24. The integral (20) may now be written

$$\int_0^1 \frac{1}{\rho}(fx)^2\,dx - \sum_{n=1}^{m} a_n b_n \int_0^1 \frac{1}{\rho}\,\xi_n(x)\,\eta_n(x)\,dx,$$

a form which shows that it decreases continually as m increases, and therefore tends to a definite limit when m increases without limit.

Also

$$a_n b_n \left\{\int_0^1 \frac{1}{\rho}\,\xi_n(x)\,\eta_n(x)\,dx\right\}^2$$

$$= \int_0^1 \frac{1}{\rho} f(x)\,\xi_n(x)\,dx \times \int_0^1 \frac{1}{\rho} f(x)\,\eta_n(x)\,dx$$

$$= \tfrac{1}{2}\int_0^1\int_0^1 \frac{1}{\rho_x\rho_t} f(x) f(t)\,\{\xi_n(x)\,\eta_n(t)+\xi_n(t)\,\eta_n(x)\}\,dx\,dt,$$

which shows that the integral (20) is equal to

$$\int_0^1 \frac{1}{\rho}(fx)^2\,dx + \frac{1}{2\iota\pi}\int_0^1\int_0^1 \frac{1}{\rho_x} f(x) f(t)\,\mathrm{F}(t,x,\mathrm{R})\,dx\,dt$$

if R is so chosen that the path (R) encloses $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Since then

$$\operatorname*{Lim}_{\mathrm{R}\to\infty}\int_0^1 \mathrm{F}(t,x,\mathrm{R})\,dt = -2\iota\pi$$

the limit to which (20) tends when R is indefinitely increased is the limit of

$$-\frac{1}{4\iota\pi}\int_0^1\int_0^1 \frac{1}{\rho_x}(fx-ft)^2\,\mathrm{F}(t,x,\mathrm{R})\,dx\,dt. \qquad (21)$$

25. From this form it is possible to prove that the limit is zero.

First, let the domain (0, 1) be divided into intervals in each of which $f(x)$ is constant. Then when x, t are in the same interval the contribution to the double integral (21) is zero. When x, t are in different intervals (x_0, x_1), (t_0, t_1), $fx-ft$ has a constant value so long as those intervals do not change, and the corresponding part of the double integral is a constant multiple of

$$\int_{x_0}^{x_1}\int_{t_0}^{t_1}\int_{(\mathrm{R})} \frac{1}{\rho_x\rho_t}\,\frac{\omega(t, x)}{\Omega}\,d\lambda\,dt\,dx.$$

In this expression take the terms of $\omega(t, x)$ separately. Taking the coefficient of K we have

$$\int_{x_0}^{x_1}\int_{t_0}^{t_1}\frac{1}{\rho_x\rho_t}\,\psi(x, 0)\,\psi(t, 1)\,dt\,dx$$

$$=\left[\frac{1}{\lambda}\Psi(x_0, 0)-\frac{1}{\lambda}\Psi(x_1, 0)+\frac{1}{\lambda}\int_{x_0}^{x_1}\sigma\psi(x, 0)\,dx\right]\left[\frac{1}{\lambda}\Psi(t_0, 1)-\frac{1}{\lambda}\Psi(t_1, 1)+\frac{1}{\lambda}\int_{t_0}^{t_1}\sigma\psi(x, 1)\,dx\right],$$

which is at most of the order of $\mathrm{R}^{-3/2}\Psi(1, 0)$ so long as $x_0 < x_1 \leq t_0 < t_1$. In this way it appears that the triple integral is at most of the order of $\mathrm{R}^{-1/2}$, and the same result can be deduced when $t_0 < t_1 \leq x_0 < x_1$ by putting $\omega(x, t)$ in the place of $\omega(t, x)$. Since the number of intervals is finite the whole expression (21) tends to zero when R is increased indefinitely, and the integral (20) can be made as small as we please when $f(x)$ is a function of the special type, constant in each of a system of sub-intervals.

Now, let $f(x)$ be unrestricted but real, $\phi(x)$ a real function of the special type, and $\chi(x)$ such an expression as

$$a_1\xi_1(x)+a_2\xi_2(x)+\ldots+a_m\xi_m(x).$$

We have

$$\int_0^1\frac{1}{\rho}\,|fx-\chi x|^2\,dx < 2\int_0^1\frac{1}{\rho}\,(fx-\phi x)^2\,dx+2\int_0^1\frac{1}{\rho}\,|\phi x-\chi x|^2\,dx,$$

and it follows from §§ 5, 6 that the first of these terms can be made arbitrarily small, if the integral of $\frac{1}{\rho}(fx)^2$ exists, by proper choice of the constant values of ϕ and of the different sub-intervals. It has now also been proved that the second term can be diminished indefinitely by a proper choice of $a_1, a_2, \ldots$ and by taking m great enough. Hence $\int_0^1\frac{1}{\rho}\,|fx-\chi x|^2\,dx$, or (20), is also arbitrarily small. Now (20) has its minimum value for any given value of m when $a_1, \ldots, a_m$ have the values assigned in § 23, and this minimum value must tend to zero when m is increased without limit.

Hence the expansion of $f(x)$, found in § 19, *may be written in the form*

$$\tfrac{1}{2}\sum_1^\infty\{a_n\xi_n(x)+b_n\eta_n(x)\},$$

where ξ_n, η_n have the meanings assigned in § 23 *and*

$$a_n=\int_0^1\frac{1}{\rho}\,f(x)\,\eta_n(x)\,dx \div \int_0^1\frac{1}{\rho}\,\xi_n(x)\,\eta_n(x)\,dx,$$

$$b_n=\int_0^1\frac{1}{\rho}\,f(x)\,\xi_n(x)\,dx \div \int_0^1\frac{1}{\rho}\,\xi_n(x)\,\eta_n(x)\,dx.$$

These values of a_n, b_n are conjugate imaginaries, and are such as to give the least possible value to

$$\int_0^1 \frac{1}{\rho} \left| f(x) - a_1\xi_1(x) - a_2\xi_2(x) - \ldots - a_m\xi_m(x) \right|^2 dx,$$

that is, to

$$\int_0^1 \frac{1}{\rho} \left\{ f(x) - \sum_1^m a_n\xi_n(x) \right\} \left\{ f(x) - \sum_1^m b_n\eta_n(x) \right\} dx.$$

The value of this integral, and therefore also that of

$$\int_0^1 \frac{1}{\rho} \left[f(x) - \sum_1^m \tfrac{1}{2} \{ a_n\xi_n(x) + b_n\eta_n(x) \} \right]^2 dx$$

tend to zero as m is increased, if $\int_0^1 \frac{1}{\rho} (fx)^2 \, dx$ *exists.*

26. The integral

$$\int_{(R)} \frac{\omega(t, x)}{\Omega} \frac{d\lambda}{\lambda - \lambda'}$$

tends to zero when $R \to \infty$ if $x \quad t$, and thus it follows that

$$\frac{\omega(t, x)}{\Omega} = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \frac{\xi_n(x)\eta_n(t) + \xi_n(t)\eta_n(x)}{2\int_0^1 \xi_n(x)\eta_n(x)\,dx}.$$

To this expression the methods of Dr. J. MERCER ('Roy. Soc. Proc.,' A, vol. 84, p. 573, and 'Phil. Trans.,' A, vol. 211, pp. 134 *ff.*) may be applied, but his idea of the bilateral limit cannot be used without some modification, since we have no reason to believe that even at a point of discontinuity where $f(x \pm 0)$ exist their mean is represented by the present expansions.

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